International Journal of Theoretical Physics, Vol. 35, No. 10, 1996

Dynamics of a Compact Hyperbolic Cosmological Model with Dustlike Matter and Radiation

Alexander Pavlov[1]

Received November 10, 1995

The classical and quantum dynamics of a compact hyperbolic cosmological model with dustlike matter and radiation are presented. The properties of homogeneity and isotropy are conserved in small regions only. We compare various other methods of quantization with our approach. The present paper is an extended and corrected version of an earlier paper.

1. INTRODUCTION

The quantization of cosmological models originates with De Witt's (1967) fundamental paper. Burlankov *et al.* (1984) developed the dynamics of a closed model of a homogeneous and isotropic universe filled with an elastic gas. The equation of state for a low-density gas corresponds to the case of dustlike matter and for a high-density gas it corresponds to the ultrarelativistic case, i.e., the equation of state contains both limiting cases of the state of matter, which had to be considered separately before.

Homogeneity and isotropy of space means that one can choose a world time such that the space metric is the same at all points at any moment of time. The properties of the curvature of such a space are determined by one variable only—its scalar curvature. So only three different cases of the space metric are possible: (i) space of positive curvature (closed model), (ii) space of negative curvature (hyperbolic model), and (iii) flat space with curvature equivalent to zero (parabolic model). Restriction of consideration to high-symmetry models allows us to treat some of the principal difficulties of the theory of gravity. We will consider the hyperbolic isotropic universe in this paper.

[1] Mathematical Simulation Institute, Udmurt State University, Izhevsk, 426034, Russia; e-mail: pavlov@matsim.udmurtia.su.

0020-7748/96/1000-2169$09.50/0

The open model possesses the undesirable feature that its spatial volume and total mass are infinite. So we consider a compactified manifold constructed of a hyperbolic polyhedron with faces pairwise identified. The homogeneity and isotropy of the matter distribution determine only a local space-time metric, but the global topology of the space-time (Fagundes, 1982, 1983, 1985, 1992a,b).

The completely connected Riemannian manifolds of constant curvature are defined as space forms (Efimov, 1978). The space forms of a negative curvature H^3/Σ can be constructed in infinite ways (Efimov, 1978), where Σ is the group of isometries of H^3. All manifolds possessing a metric of a given constant curvature have the same geometry in the small. Each manifold assumes congruent, in the sense of its geometry, movements of sufficiently small elements along it, but metrized manifolds of different topological types, on the whole, possess different geometries. For our case it is not important what method of compactification is taken. A unique demand for a scheme of compactification of a manifold is to reduce the surface divergent terms in the Lagrangian (De Witt, 1967) of a gravitational field. In the final formulas topological peculiarities of the compactified manifold are not essential. The formulas contain the value of its volume only.

The classical gravitational equations of the model admit exact solutions. Because the theory of relativity is formulated in an extended phase space, in the Hamiltonian language of description we carry out a Hamiltonian reduction after some canonical transformation. Then in the chosen system of reference we calculate the total energy of the system. However, addendums of the energy do not have the traditional sense for the mixing of terms of gravitational and matter origin.

Then we develop a quantum description of the behavior of the model considering transitions of the universe through a collapse point. The quantum problem is also exactly solvable. An examination of the correspondence principle leads to the Ehrenfest theorem. The quantum average mean of a scalar factor of the universe and its canonical conjugate variable follow by classical laws. As to their dispersions, the wave packets diffuse in time.

The final section compares various popular methods of quantization.

2. CLASSICAL DYNAMICS OF THE MODEL

According to the principles of general relativity, the metric of a four-dimensional continuum is defined locally by the state of matter. However, the space-time structure can have an extremely complicated character in view of the nonhomogeneous distribution of matter. In studying the structure of a space-time manifold one may consider matter as quite homogeneously distributed for sufficiently extended space scales. Consequently matter can be described by a slowly changing function. The first quadratic form of an open model can be taken as

$$ds^2 = a^2(\eta)[-N^2\, d\eta^2 + d\chi^2 + \sinh^2\chi\, (d\theta^2 + \sin^2\theta\, d\varphi^2)] \tag{2.1}$$

where a is the pseudosphere radius at the moment η; N is a lapse-like function; and η, χ, θ, and φ are dimensionless coordinates of the pseudosphere. The pseudosphere with radius a can be completely embedded into pseudo-Euclidean space. The imbedded manifold is a hyperboloid of two sheets. It is disposed inside the light cone. The upper sheet of the hyperboloid corresponds to the case of $a > 0$ (world) and the lower one to the case of $a < 0$ (antiworld). On this hypersurface the metric of Lobachevsky is induced.

In this paper we investigate a universe filled with masses (hazinesses, cosmic dust) and radiation with the equation of state

$$\epsilon = c^2\rho + \frac{3}{4}(A\rho^2)^{2/3} \tag{2.2}$$

where c is the light speed, ϵ is the volume density of the gas energy, ρ is the volume density of the mass, and A is expressed through the radiation energy shown below.

The second law of thermodynamics can be obtained as a result of so-called laws of conservation in the gravitation. For isentropic processes to which we are restricted here, we have

$$d\left(\frac{\epsilon}{\rho}\right) = -p\left(\frac{1}{\rho}\right) \tag{2.3}$$

where ρ is the pressure. Then, introducing the mass density of the enthalpy w

$$w = (\epsilon + p)/\rho \tag{2.4}$$

one can rewrite the second law:

$$dw = \frac{1}{\rho}\, dp \tag{2.5}$$

The relation between differentials of the right and left sides of equation (2.4), taking into account (2.5), gives us one more form of the second law:

$$d\epsilon = w\, d\rho \tag{2.6}$$

Using these formulas, one gets the expression for the pressure of the gas through its enthalpy density:

$$w = \frac{d\epsilon}{d\rho} = c^2 + (\rho A^2)^{1/3}, \qquad \rho = \frac{(w - c^2)^3}{A^2}$$

$$p = \int \rho\, dw = \frac{(w - c^2)^4}{4A^2} \tag{2.7}$$

Notice that the equation of state for a low-density gas corresponds to the case of dustlike matter $\epsilon = c^2\rho$, as is seen from (2.7); for a high-density gas it corresponds to the ultrarelativistic case $\epsilon = 3p$, because for this case

$$w = (\rho A^2)^{1/3}, \qquad p = w^4/(4A^2)$$

and excluding w from (2.7), one finds the connection between ρ and p:

$$p = \frac{1}{4}(A\rho^2)^{2/3} \tag{2.8}$$

Hence we have the relation

$$\epsilon = 3p \tag{2.9}$$

So the model of the universe investigated is really filled with massive particles and radiation.

The action functional of the gas is a scalar and defined through $p(w)$ as

$$S_m = \int d^4x\, p(w)\sqrt{-g} \tag{2.10}$$

where g is the determinant of the metric tensor. Let us describe the enthalpy density w of the gas as the derivative of the potential function $\sigma(\eta)$ with respect to the intrinsic time τ (Burlankov *et al.*, 1984):

$$w = \frac{d\sigma}{d\tau} \tag{2.11}$$

It can be shown that the momentum-energy tensor corresponds to that of an ideal liquid. An adiabatic without whirling motion of an elastic gas in general relativity can be described by using a potential function $\sigma(x^\mu)$. It defines the enthalpy density of a gas

$$w = \left(g^{\mu\nu}\frac{\partial\sigma}{\partial x^\mu}\frac{\partial\sigma}{\partial x^\nu}\right)^{1/2} \tag{2.12}$$

The variation of S_m by σ leads to equations of motion

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,\frac{dp}{dw}\,g^{\mu\nu}\frac{\partial\sigma}{\partial x^\nu}\frac{1}{w}\right) = 0 \tag{2.13}$$

As $dp/dw = \rho$ and the vector field

$$u^\mu = g^{\mu\nu}\frac{\partial\sigma}{\partial x^\nu}\frac{1}{w} \tag{2.14}$$

obeys

$$g_{\mu\nu}u^\mu u^\nu = 1 \tag{2.15}$$

(consequently, u^μ is the 4-velocity of the gas), so equation (2.13) is the conservation law for the mass of the gas, meaning that

$$\nabla_\mu(\rho u^\mu) = 0 \tag{2.16}$$

The energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\partial(p\sqrt{-g})}{\partial g^{\mu\nu}} = \frac{dp}{dw}\frac{1}{w}\frac{\partial\sigma}{\partial x^\mu}\frac{\partial\sigma}{\partial x^\nu} - pg_{\mu\nu} \\ &= \rho w u_\mu u_\nu - pg_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - pg_{\mu\nu} \end{aligned} \tag{2.17}$$

corresponds to that of an ideal liquid. In the isotropic case we get (2.11).

Now, let us obtain the Hilbert functional from the Arnowitt–Deser–Misner (ADM) approach, which is based on separating the 4-dimensional space-time into time and space. It is developed for Hamiltonian systems with constraints.

The space metric tensor g_{ik} is

$$(g_{ik}) = a^2(\eta)\,\mathrm{diag}(1, \sinh^2\chi, \sinh^2\chi\,\sin^2\theta) \tag{2.18}$$

Using it, we obtain the external curvature tensor K_{ik} of the space:

$$K_{ik} = \frac{1}{(2aN)}\dot{g}_{ik} \tag{2.19}$$

Then, by using (2.18) and (2.19), we get the Ricci scalar density

$$Nag^{1/2}(K_{ik}K^{ik} - K^2 + R) = -3\sin^2\theta\left(\frac{\dot{a}^2}{2N} + N\frac{a^2}{2}\right) \tag{2.20}$$

We shall integrate this expression to obtain the action by space coordinates over the finite space region. It is isometric to a fundamental polyhedron. A closed universe is preferable to an open one. If we consider problems related to a planetary system, we choose boundary conditions in such a way that there is a coordinate system where all gravitational potentials are Minkowskian at spatial infinity. But *a priori* it is not obvious that we have the right to use the same boundary conditions if we consider the universe as a whole. Einstein suggested a way to avoid boundary conditions: the question could be avoided if a world continuum is closed with reference to space and has a finite 3-dimensional volume.

Setting $c = 1$, $G = 3/(16\pi)$, and a dimensionless space volume of unit scale $V = 1$, we get the resulting action (the Hilbert functional plus the matter action):

$$S = \int_0^{\eta_1} d\eta\left[-\frac{\dot{a}^2}{2N} - N\frac{a^2}{2} + \frac{N}{4A^2}\left(\frac{\dot{\sigma}}{N} - a\right)^4\right] \tag{2.21}$$

The description of the gauge theory dynamics follows the Hamiltonian approach. The Hamiltonian formalism is introduced by the Legendre transformation. We get an extended phase space with Poisson brackets. The momenta canonically conjugate to the variables a and σ, respectively, are

$$p_a = -\frac{\dot{a}}{N}, \qquad p_\sigma = \frac{1}{A^2}\left(\frac{\dot{\sigma}}{N} - a\right)^3$$

The Hamiltonian H is

$$H = N\left[-\frac{1}{2}p_a^2 + \frac{1}{2}(a + p_\sigma)^2 - \frac{1}{2}p_\sigma^2 + \frac{3}{4}(Ap_\sigma^2)^{2/3}\right] = N\varphi_0 \qquad (2.22)$$

where φ_0 is the Hamiltonian constraint.

The generalized Poincaré–Cartan differential form is $\omega^1 = p_a\,da + p_\sigma\,d\sigma - N\varphi_0\,d\eta$. From the closure conditions of the 1-form and by force of topological triviality of the phase space one gets the Hamiltonian equations on the constraint hypersurface:

$$\frac{da}{d\eta} = -Np_a, \qquad p_\sigma = \text{const}$$

$$\frac{dp_a}{d\eta} = -N(a + p_\sigma), \qquad \frac{d\sigma}{d\eta} = N[a + (A^2p_\sigma)^{1/3}] = aNw \qquad (2.23)$$

$$\frac{1}{2}p_a^2 - \frac{1}{2}(a + p_\sigma)^2 + \frac{1}{2}p_\sigma^2 - \frac{3}{4}(Ap_\sigma^2)^{2/3} = 0$$

We obtain the enthalpy density w from (2.23) immediately

$$w = 1 + \frac{1}{a}(A^2p_\sigma)^{1/3} \qquad (2.24)$$

So the thermodynamic characteristics of the gas have a singularity at the point $a = 0$:

$$\rho = \frac{p_\sigma}{A^3}, \qquad p = \frac{(Ap_\sigma^2)^{2/3}}{4a^4}$$

$$\epsilon = \frac{p_\sigma}{a^3} + \frac{3}{4}\frac{(Ap_\sigma^2)^{2/3}}{a^4} \qquad (2.25)$$

[compare with Schmutzer *et al.* (1980)]. Hence, as follows from (2.25), the numerical value for p_σ gives a dust mass, $p_\sigma = M$, which is conserved in time; the parameter A is connected with the radiation energy.

The Friedmann equation in the form of conservation of energy follows from the system (2.23):

$$\frac{1}{2}\left(\frac{da}{d\eta}\right)^2 - \frac{1}{2}(a + M)^2 = E \tag{2.26}$$

where the total energy E (together with the gravitational one)

$$E = -\frac{1}{2}M^2 + \frac{3}{4}(AM^2)^{2/3} \tag{2.27}$$

is conserved and is not added as the sum of the mass energy, radiation energy, and gravitational interaction energy. The problem is reduced to the effective mechanical one of a particle in a quadratic potential.

It is necessary to say something about the problem of energy in the theory of gravity. The question of the definition of the main integrals of the motion—energy, momentum, and angular momentum—was raised at once after the creation of the theory of gravitation by Einstein in 1915 and was solved by him mainly in 1918: the term "general relativity" which was given by the author to his creation relates not to dynamics, but to a definition of the gauge group of symmetry. The theory is formed in a covariant form relative to a group of coordinate transformations. But if one wishes to consider the dynamics, it is necessary to go over to a Hamiltonian noncovariant formalism. And here, strange though it may seem, it is just the gauge invariance in the description of the theory (which has led to the unified theory of electroweak interactions and provided a clue in superstring theories trying to unify all physical interactions) that puts obstacles to the study of the dynamics (to found dynamical characteristics and then to quantize the theory). It is necessary to go from the canonical Lagrange formulation of the theory to a noncovariant Hamiltonian one, i.e., to find an adequate observer in whose system of reference we are to determine the dynamics and calculate the conserved characteristics.

The concept of energy (and the other main integrals as well) of a gravitational field interacting with a system of mass and fields of matter is introduced in traditional theoretical physics only in the case when space-time at spatial infinity is reduced to Minkowski space. A dynamical shift along the time is defined relative to an observer sufficiently far from gravitating masses. Then the energy in the observer's system of reference is calculated by using the Noëther theorem. The existence of ten integrals of motion is connected with the isotropy of the Minkowski space on the background of which the system is being studied. In contrast, in cosmology there is no such observer gazing at the universe from outside and keeping an eye on it and using his or her own watch. So the usual way of calculating is not appropriate for cosmological problems. But the worlds described by Einstein's equations are much more interesting: among them there are topological nontrivial space-times and black holes. According to Einstein, laws of nature must be formulated in a form independent of the choice of a reference system because they are formally equivalent. If we choose a system, the theory of physical phenomena must be described entirely in terms which belong only to this system. Mental glances of arbitrary contemplators in classical gravity at the world are only a method of description, but not a necessary element of physical laws. The essence of the principle of general relativity is contained in this "democracy." But in each concrete problem it is convenient to take a system of observation where the analysis and solution of the problem are simplified. The use of Minkowski space as a certain fundamental one for an analysis of all gravitational problems is not logical from Einstein's point of view. As we mentioned, one can obtain in a standard way conservation laws for an energy-momentum and an angular momentum of a substance together with gravitational fields on the bimetric basis only in an asymptotically Minkowski space. The flat space having a 4-parameter group of transitions and a 6-parameter group of rotations is formally equivalent to a Riemannian one. And the space is convenient if we consider simple island-type objects only.

In order to ask how to define the energy of the universe, it is necessary to introduce its conjugate characteristic associated with change—the time. But there is no such abstract clock hanging over the universe and counting an abstract time. Changes which occur in it and which we are to describe lead us to build a characteristic relative to which a development takes place. And this time is not derived from something external, but from inner characteristics of the universe. Thus one can consider how to obtain the energy conserved in this time.

There is no general method of simplification of the Lagrange function nor systematic approach to obtaining cyclic variables. In contrast, in Hamiltonian mechanics there is a method for getting cyclic variables and a simplification of the Hamiltonian function. This approach reduces the problem of the integration of motion equations to obtaining a generation function of some transformation. This method of coordinate transformation is based on a different approach as compared with the problem of direct integration. It is important that under transformations there are conserved canonical equations, i.e., the Poincaré–Cartan form is conserved. So the dynamical problem is reduced to one of the theory of groups.

Let us perform the canonical transformations which keep ω^1 closed, denoting

$$x \equiv a + p_\sigma, \qquad p_x \equiv p_a$$

We have

$$\omega^1 = p_x\, dx + p_\sigma\, d(\sigma + p_x) + N\varphi_0\, d\eta - d(p_x p_\sigma)$$

$$= p_x\, dx + p_\sigma\, d\left(\frac{\sigma + p_x}{p_\sigma - (A^2 p_\sigma)^{1/3}}\right)[p_\sigma - (A^2 p_\sigma)^{1/3}]$$

$$+ \frac{(\sigma + p_x)[p_\sigma - (1/3)(A^2 p_\sigma)^{1/3}]}{p_\sigma - (A^2 p_\sigma)^{1/3}}\, dp_\sigma + N\varphi_0\, d\eta - d(p_x p_\sigma) \quad (2.28)$$

Then one constructs the "inner" time t from the coordinates of an expanded phase space:

$$t = \frac{\sigma + p_x}{p_\sigma - (A^2 p_\sigma)^{1/3}} \quad (2.29)$$

which coincides with η, $dt/d\eta = 1$ (so also a dimensionless characteristic), and rewrites (2.28) using it:

$$\omega^1 = p_x\, dx + [p_\sigma^2 - (Ap_\sigma^2)^{2/3}]\, dt + t\left[p_\sigma - \frac{1}{3}(A^2 p_\sigma)^{1/3}\right] dp_\sigma$$

$$- d(p_x p_\sigma) + N\varphi_0\, d\eta$$

$$= p_x\, dx - \left[-\frac{1}{2} p_\sigma^2 + \frac{3}{4}(Ap_\sigma^2)^{2/3}\right] dt + N\varphi_0\, d\eta$$

$$+ d\left[t\left(\frac{1}{2} p_\sigma^2 - \frac{1}{4}(Ap_\sigma^2)^{2/3}\right) - p_x p_\sigma\right]$$

$$= p_x\, dx - p_t\, dt + d\omega^0 + N\varphi_0\, d\eta \quad (2.30)$$

So the generating function $\omega^0 = p_t t + p_\sigma \sigma$ as a function of old and new variables is

$$\omega^0(a, \sigma; x, t) = -\frac{1}{2}(x - a)^2 t + \frac{3}{4} A^{2/3}(x - a)^{4/3}\, t$$

$$+ (x - a)\sigma \quad (2.31)$$

The constraint is the generator of the infinitesimal redefinition of the time η by means, and hence it transforms the dynamical variations. But this transformation does not change the dynamical state of the system. If we solve the constraint equation

$$-\frac{1}{2} p_\sigma^2 + \frac{3}{4}(Ap_\sigma^2)^{2/3} = \frac{1}{2} p_x^2 - \frac{1}{2} x^2 \quad (2.32)$$

and reduce the 1-form (2.31), then we get

$$\omega^1 = p\,dx - H(p, x)\,dt \tag{2.33}$$

with the Hamiltonian $H(p, x)$

$$H(p, x) = \frac{1}{2}p^2 - \frac{1}{2}x^2 \tag{2.34}$$

or, in the initial system of units,

$$H(p, x) = \left(\frac{16\pi G}{3c^3V}\right)\frac{p^2}{2} - \left(\frac{3c^3V}{16\pi G}\right)\frac{x^2}{2}$$

This makes it clear that we are dealing with a conservative system having energy E:

$$E = -\frac{1}{2}M^2 + \frac{3}{4}(AM^2)^{2/3} \tag{2.35}$$

Unfortunately, there are no systematic rules for finding the most successful canonical transformation and for reducing the phase space of an arbitrary Hamiltonian system. The choice of these new coordinates is an art.

The Hamiltonian $H(p, x)$ generates the phase flow. The phase "portrait" of the two-dimensional system studied here has an unstable singular saddle point. The phase curves correspond to the motion of a particle in the potential $U(x) = -(1/2)x^2$ of the harmonic oscillator with negative stiffness. This effective problem describes the dynamics of variation of the compactified pseudosphere with curvature radius a in time t. The effect of the nonregularity of the behavior of the system due to the "turned over" harmonic potential will show up on the quantum level of description.

The curvature radius $a(t)$ as well as the intrinsic time τ increase exponentially with the time t. This agrees with the solutions of the Friedmann model. This system dynamically is unsteady. An arbitrary small perturbation of initial conditions leads to arbitrarily large deviations of the phase trajectory from the nonperturbative one situation. This will also show up on the quantum level of description of the system. The total energy E conserved in time also is not added as the sum of the mass energy, radiation energy, and gravitational interaction energy. We restrict consideration of energies to the interval $-\frac{1}{2}M^2 < E < \frac{1}{4}M^2$. The lower inequality ($A = 0$) follows from the demand of positive gas pressure, the upper one ($M = A$) from a univalent canonical map.

Let us analyze now the solutions of the Friedmann equations in detail, considering all solutions, including nonphysical ones.

For $E > 0$, i.e., $M < (\frac{3}{2})^{3/2}A$, we can say that radiation "prevails" over matter. The solutions $a(t)$ and $\tau(t)$ in parametrized form are

$$a(t) = -M + \sqrt{2E}\sinh(t), \qquad \tau(t) = -Mt + \sqrt{2E}\cosh(t) \quad (2.36)$$

and the dependence between the radius of curvature and intrinsic time is

$$\tau(a) = [2E + (a + M)^2]^{1/2} - M \operatorname{arcsinh}[(a + M)/\sqrt{2E}] \quad (2.37)$$

The transition of the scale factor $a(t)$ from positive quantities to negative ones may be interpreted as the collapse of the world ($a > 0$), transition through a singularity ($a = 0$), and big bang creation of an antiworld ($a < 0$). The space-time metric is invariant relative to a change of the sign of the scale factor, but the flow of the proper time τ of the universe suffers T *reflection.* As to space, it "turns inside out" (*P reflection*), as the embedding coordinates into six-dimensional pseudo-Euclidean space change sign. If the sign of a changes, the sign of ρ also changes, as $M = \rho a^3$, and M is the integral of the motion, which leads us to interpret this event in terms of relativistic quantum field theory as *C transformation.* Thus *CPT transformation* takes place at the singular point of space-time, as was already suggested (Burlankov *et al.*, 1984).

For the case when the universe is filled with radiation only, $M = 0$, $A = \alpha^{3/2}/M^2$ (α is a parameter which characterizes the radiation, $E = (3/4)\alpha$, $\epsilon = 3\alpha/(4a^4)$], we have

$$\begin{aligned} a(t) &= \sqrt{2E}\sinh(t) \\ \tau(t) &= \sqrt{2E}\cosh(t) \\ \tau^2 - a^2 &= 2E \end{aligned} \quad (2.38)$$

The dependence $\tau = \tau(a)$ in the vicinity of $a = 0$ is quadratic: $\tau \sim a^2$.

For the second case, $E = 0$, $M = (\frac{3}{2})^{3/2}A$, a partial solution of the Friedmann equation is

$$\begin{aligned} a(t) &= M[\exp(t) - 1] \\ \tau(t) &= M[\exp(t) - t + 1] \\ \tau(a) &= a - M\log[(a + M)/M] \end{aligned} \quad (2.39)$$

Finally, we have the most interesting and realistic case: the dust "prevails" over radiation. In this case when the dust is "mixed" with radiation, $-(1/2)M^2 < E < 0$, the solutions of the Friedmann equation are

$$\begin{aligned} a(t) &= -M + (2|E|)^{1/2}\cosh(t) \\ \tau(t) &= -Mt + (2|E|)^{1/2}\sinh(t) \\ \pm\tau(a) &= [(a + M)^2 - 2|E|]^{1/2} - MAr\cosh[(a + M)/(2|E|)^{1/2}] \end{aligned} \quad (2.40)$$

We see that a cycle results, the possibility of which was discussed in an article about a closed universe by Burlankov *et al.* (1984). At the moment τ_1,

$$\tau_1 = -[(M^2 - 2|E|)^{1/2} - MAr\cosh(M)/(2|E|)^{1/2}] \tag{2.41}$$

a world–antiworld pair is created. At the moment $\tau_2 = -\tau_1$ the annihilation of the antiworld with a primary compressed world takes place. The world born at the time τ_1 extends freely. Intrinsic clocks are not very convenient because they change their rates. This interpretation is natural, for these solutions were found without any major modifications of general relativity such as, for example, the introduction of an infinite to high wall as by De Witt (1967).

If the radiation is absent, $M = (2|E|)^{1/2}$, $A \to 0$, the solutions are

$$\begin{aligned} a(t) &= M[\cosh(t) - 1] \\ \tau(t) &= M[\sinh(t) - t] \\ \pm\tau(a) &= [a(a + 2M)]^{1/2} - MAr\cosh[(a + M)/M] \end{aligned} \tag{2.42}$$

If t is small, $\tau = \frac{1}{3}(2a^3/M)^{1/2}$.

Finally, if $E < -(1/2)M^2$ (the imaginary case $p < 0$), the world does not transit through the singularity.

3. QUANTUM DYNAMICS OF THE MODEL

We consider transitions of the universe through the singularity, so we must develop a quantum description of the behavior of the model in the vicinity of the singularity when the scale of the phenomena is the Planck scale. Many methods of quantization are available, and lead to inequivalent results. No unique successful scheme for the quantization of gravity has been elaborated. In this paper we use the reduced phase-space quantization (RPSQ) "first reduce and then quantize" method (or Arnowitt–Deser–Misner, ADM, method). But first we simplify the classical system by using canonical transformations (see Section 2). The canonical transformations plus RPSQ permits us to solve the problem completely. The consistency of the quantum mechanical results with the classical representation will be shown by proving the Ehrenfest theorem.

We can obtain the quantum equation if we substitute all dynamical variables in the classical equation by differential operators which act in the linear vector space of states (Dirac mapping). So we obtain a differential equation with the standard interpretation of its solutions through a Schrödinger-like equation. Let us put $\hbar = 1$, i.e., we choose the Planck system of units:

$$i \frac{\partial}{\partial t} \psi = \frac{1}{2} \frac{\partial^2}{\partial x^2} \psi + \frac{1}{2} x^2 \psi \quad (3.1)$$

The wave function satisfies the Schrödinger equation for the oscillator with an imaginary frequency. We can rewrite the Hamiltonian operator using the initial units

$$\hat{H}(\hat{p}, x) = \frac{l_p^2}{\hbar V} \frac{\hat{p}^2}{2} - \frac{\hbar V}{l_p^2} \frac{x^2}{2}$$

where the Planckian length is

$$l_p = \left(\frac{16\pi G\hbar}{3c^3}\right)^{1/2}$$

Let us show how the quantum state space can be built. The stationary states are

$$\psi(t, x) = e^{iEt}\psi(x) \quad (3.2)$$

Now we have to solve the eigenvalue problem on a restricted function $\psi(x)$:

$$\left(\frac{d^2}{dx^2} + x^2 + E\right)\psi(x) = 0 \quad (3.3)$$

This is a second-order equation with the irregular point $x = \infty$. We represent the ψ-function in the form

$$\psi(x) \sim e^{\pm ix^2/2}\psi(x) \quad (3.4)$$

Then the differential equation becomes

$$\psi''(x) - 2\lambda x\psi'(x) + (k^2 - \lambda)\psi(x) = 0 \quad (3.5)$$

where $\lambda = \mp i$, $k^2 = 2E$. Introducing a new variable $y = \lambda x^2$, we obtain the Gauss equation (Smirnov, 1974)

$$y\psi''(y) + \left(\frac{1}{2} - y\right)\psi'(y) + \left(\frac{k^2}{4\lambda} - \frac{1}{4}\right)\psi(y) = 0 \quad (3.6)$$

where derivatives are taken with respect to the variable y. Setting $a = \frac{1}{4} - k^2/(4\lambda) = \frac{1}{4} \mp iE/2$, we express the general solution of the equation in terms of degenerate hypergeometric functions $F(a, b; y)$ (Smirnov, 1974):

$$\psi(y) = AF\left(a, \frac{1}{2}; y\right) + By^{1/2}F\left(a + \frac{1}{2}, \frac{3}{2}; y\right) \quad (3.7)$$

The hypergeometric function $F(a, b; y)$ is regular at the point $y = 0$, and one can show that

$$F(a, b; y) = 1 + \frac{a}{b}\frac{y}{1!} + \frac{a(a+1)}{b(b+1)}\frac{y^2}{2!} + \cdots \tag{3.8}$$

The point $y = \infty$ is an irregular singular point. As $y \to \infty$, both terms in (3.7) diverge as $e^y y^{a-1/2}$ (Smirnov, 1974), or, returning to the variable x,

$$\psi(x) \sim \exp\left(\frac{\mp ix^2}{2}\right) \frac{[(\mp i)^{1/2}|x|]^{\mp iE}}{[(\mp i)^{1/2}|x|]^{1/2}} \tag{3.9}$$

To normalize the functions $\psi_E(x)$, it is sufficient to use their asymptotic expressions (Landau and Lifshitz, 1974). Then, using (3.9), we get

$$\begin{aligned} &\int_{-\infty}^{+\infty} dx\, \psi_{E'}^{*}(x)\psi_E(x) \\ &= \int_{-\infty}^{+\infty} \frac{dx}{|x|} |x|^{\pm i(E'-E)} \\ &= \int_{-\infty}^{+\infty} d\log(x) \exp[\pm i(E'-E)\log|x|] \\ &= 2\pi\delta(E'-E) \end{aligned} \tag{3.10}$$

The eigenfunctions of the Hamilton operator are

$$\begin{aligned} \psi_E(t, x) &= \frac{1}{2\sqrt{2\pi}} \exp\left(\frac{ix^2}{2}\right) F\left(\frac{1}{4} - i\frac{E}{2}, \frac{1}{2}; -ix^2\right) \\ \psi_E(t, x) &= \frac{\sqrt{-i}}{2\sqrt{2\pi}} \exp\left(\frac{ix^2}{2}\right) xF\left(\frac{3}{4} - i\frac{E}{2}; -ix^2\right) \end{aligned} \tag{3.11}$$

This basis is orthonormal and complete. The spectrum of the states is continuous.

Now we consider the dynamics of the quantum average means of wave packets. Since the Hamiltonian is a polynomial of the second order in x and p, their quantum average means follow the classical trajectory,

$$\begin{aligned} \frac{d}{dt}\langle\hat{x}\rangle &= \left\langle\frac{\partial\hat{H}}{\partial p}\right\rangle = \langle\hat{p}\rangle \\ \frac{d}{dt}\langle\hat{p}\rangle &= -\left\langle\frac{\partial\hat{H}}{\partial x}\right\rangle = \langle\hat{x}\rangle \end{aligned} \tag{3.12}$$

The spectra of the scale operator $\hat{a}$ and the intrinsic time $\hat{\tau}$ are continuous and occupy all the real line. Their average means

$$\langle \hat{a} \rangle = \langle \hat{x} \rangle - \langle \hat{p}_\sigma \rangle$$
$$\langle \hat{\tau} \rangle = -\langle \hat{p}_\sigma \rangle t + \langle \hat{p}_x \rangle \tag{3.13}$$

change according to the classical formulas. Note that the operator of intrinsic time is obtained by the following definition: $d\tau = a\, dt = (x - p_\sigma)\, dt = dp - p_\sigma\, dt$, so $\tau = p_x - p_\sigma t$.

We can conclude that the Ehrenfest theorem holds in our model. Now, we investigate the evolution of dispersions of $\hat{x}$ and $\hat{p}$ of the antioscillator,

$$\chi = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2$$
$$\omega = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \tag{3.14}$$

Their evolutions can be obtained from the Schrödinger equation:

$$i \frac{d}{dt} \chi = \langle [\hat{q}^2 - \langle \hat{q} \rangle^2, \hat{H}] \rangle$$
$$i \frac{d}{dt} \omega = \langle [\hat{p}^2 - \langle p \rangle^2, \hat{H}] \rangle \tag{3.15}$$

For the Hamiltonian $H = p^2/2 + V(q)$, we have

$$\frac{d}{dt} \chi = \langle pq + qp \rangle - 2\langle p \rangle \langle q \rangle \tag{3.16}$$

which can be differentiated one more time:

$$\frac{d^2}{dt^2} \chi = 2\omega - (\langle V'q + qV' \rangle - 2\langle q \rangle \langle V' \rangle) \tag{3.17}$$

We introduce the variable ϵ:

$$\epsilon \equiv \langle H \rangle - E_{cl} = \frac{1}{2} \omega + \langle V(q) \rangle - V(\langle q \rangle) \tag{3.18}$$

and get

$$\epsilon = \frac{1}{2} \omega \pm \frac{1}{2} \chi = \text{const} \tag{3.19}$$

The upper sign is for the oscillator, and the lower one is for the antioscillator. As a result we obtain the differential equation for the antioscillator case:

$$\frac{d^2}{dt^2} \chi - 4\chi - 4\epsilon = 0 \tag{3.20}$$

Then the general solutions of these equations are

$$\chi = C_1 e^{2t} + C_2 e^{-2t} - \epsilon$$
$$\omega = 2C_1 e^{2t} + 2C_2 e^{-2t} + \epsilon \tag{3.21}$$

where C_1 and C_2 are some constants. The dispersions increase exponentially in time. We stress that the exponents are expressed through time t only.

To compare the motion of the quantum packet with the motion of the classical particle, it is necessary, first, that the average means of the coordinate and the momentum of the packet should follow the classical mechanical laws, which hold in our case, and second, that the dimension of the packet in the minisuperspace must be small enough at any time moment. But, in our case, because of the exponential increase in time of the dispersion, diffusion of the packet occurs. This fast reduction is the consequence of the dynamical nonstability of the classical solutions. The other principal correlation of quantum mechanics—between energy and intrinsic time—also holds in view of the exponential increase of the dispersion of the time operator.

Proceeding with the analogy with quantum mechanics, one can define the "barrier factor" $D(E)$ for a particle penetrating through the potential barrier $U(x) = -x^2/2$ in the minisuperspace, since the quantum description of physical values has a statistical character.

The Wheeler superspace, as we mentioned, in the general case is built as a factor of Riem(M) (the space, every point of which is some Riemann metric, and some state of matter over the base manifold M) by the group of coordinate diffeomorphisms Diff(M): Riem(M)/Diff(M). In our case it is reduced to one-dimensional space.

We choose the asymptotic expressions, as in standard quantum mechanics (Landau and Lifshitz, 1974), for the wave functions in this minisuperspace:

$$\psi(x) = Bx^{iE-1/2}\exp\left(\frac{i}{2}x^2\right), \qquad x \to +\infty$$
$$\psi(x) = (-x)^{-iE-1/2}\exp\left(-\frac{i}{2}x^2\right) + A(-x)^{iE-1/2}\exp\left(\frac{i}{2}x^2\right), \qquad x \to -\infty \tag{3.22}$$

where A and B are scattering amplitudes. The first term in (3.22) describes the incident wave, the second term the reflected one. The direction of spreading of the wave is that of increasing phase. The constraint between $A(E)$ and $B(E)$ can be found proceeding from quasiclassical ideas.

The asymptotic expressions for ψ hold at a sufficiently remote region of the complex variable x, where the quasiclassical situation holds (Landau and Lifshitz, 1974). Then the coefficient $D(E)$ is

$$D = |B|^2 = \frac{1}{1 + \exp(-2\pi E)} \tag{3.23}$$

To conclude: Quantum effects do not exclude the collapse of the model universe. The "world" reaches the singularity, transits it, and then is born as an "antiworld," in the sense mentioned. The quantum approach carries the probability analysis into the studied problem of the collapse, but does not forbid its possibility.

4. COMPARISON OF VARIOUS METHODS OF QUANTIZATION

We compare our method of quantization (a canonical transformation and a Hamiltonian reduction of a classical system before the canonical quantization) and other modern approaches (Guven and Ryan, 1992).

Dirac quantization is based on replacing the Hamiltonian constraint (2.22) by a quantum operator (Dirac mapping) which annihilates physical states of the system

$$\left[-\frac{1}{2}\hat{p}_a^2 + \frac{1}{2}(a + \hat{p}_\sigma^2)^2 - \frac{1}{2}\hat{p}_\sigma^2 + \frac{3}{4}(A\hat{p}_\sigma^2)^{2/3}\right]|\psi\rangle = 0 \tag{4.1}$$

This is, in fact, the Wheeler–De Witt equation. It is evident either from the different identification of time (σ plays the role of time) and the type of differential operator that the Dirac quantization differs from our approach. The Born interpretation of the ψ-function is lost because there is no standard equation of continuity for some conserved current. So it is rather difficult to invoke this in solving this differential equation.

The essence of the ADM quantization is in resolving the constraint equation $\phi_0(a, p_a, p_\sigma)$ on the classical level of description with respect to p_σ, $p_\sigma = p_\sigma(a, p_a)$, and in imposing on the ψ-function an operator restriction:

$$(\hat{p}_\sigma - \hat{p}_\sigma(a, \hat{p}_a))|\psi\rangle = 0 \tag{4.2}$$

The algebraic constraint is to be solved analytically because it is a fourth-order algebraic equation in p_σ. Unfortunately, since the solution is expressed by means of a radical, it is difficult to treat. In principle, the ADM solutions and Dirac solutions differ. Both the Dirac approach and the ADM approach do not yield the standard quantum mechanical interpretation of the wave function.

Also, in both cases the notorious operator-ordering problem (between $\hat{a}$ and $\hat{p}_a$) holds.

Another method of quantization developed for constrained system is the Faddeev–Popov approach based on path integrals. The Green function for the antioscillator can be written by using a functional integral (Slavnov and Faddeev, 1988)

$$\langle x'', t'' | x', t' \rangle = \int Dp_x\, Dx \exp\left\{ i \int_{t'}^{t''} [p_x\, dx - H(p_x, x)\, dt] \right\} \tag{4.3}$$

where $H(p_x, x)$ is the Hamiltonian of the antioscillator and at the ends of the closed time interval the coordinates are fixed: $x(t_n) = x''$, $x(t_0) = x'$.

The time interval has been divided into N equal intervals Δt. The integrations over x are performed at points $t_k = t_0 + k\Delta t$, $k = 1, \ldots, N-1$, and those over the momentum are performed at midpoints $t_{j+1/2} = t_0 + (k + 1/2)\Delta t$, $k = 0, \ldots, N-1$. The x is fixed and its conjugate momentum is free at the initial and final points. We have

$$\begin{aligned}
&\langle x'', t'' | x', t' \rangle \\
&= \int \prod_{j=0}^{n-1} \frac{dp_x}{2\pi} (t_{j+1/2}) \prod_{k=1}^{n-1} dx(t_k) \\
&\quad \times \exp i \sum_{l=0}^{n-1} \left\{ p_x(t_{l+1/2})(x_{l+1} - x_l) - \left[\frac{1}{2} p_x^2(t_{l+1/2}) - \frac{1}{2} x^2(t_l) \right] \Delta t \right\}
\end{aligned} \tag{4.4}$$

The next step in the Faddeev–Popov method is to add an additional degree of freedom $y \equiv t$, and a constraint $\varphi_0 = p_y + H(p_x, x)$ at intermediate points of the scheme between the boundaries, and consider the evolution of the system in the parameter η:

$$\begin{aligned}
&\langle x'', t'' | x', t' \rangle \\
&= \int \prod_{j=0}^{n-1} \frac{dp_x(\eta_{j+1/2})}{2\pi} \frac{dp_y(\eta_{j+1/2})}{2\pi} \\
&\quad \times \prod_{k=1}^{n-1} dx(\eta_k)\, dy(\eta_k) \prod_{l=0}^{n-1} d\lambda(\eta_{l+1/2}) \Delta\eta \\
&\quad \times \prod_{k=1}^{n-1} \delta(y - [t_0 + \Delta\eta k]) \\
&\quad \times \exp i \sum_{m=0}^{n-1} [p_x(\eta_{m+1/2})(x_{m+1} - x_m) + p_y(\eta_{m+1/2})(y_{m+1} - y_m)] \\
&\quad \times \exp -i \sum_{n=0}^{n-1} \lambda(\eta_{n+1/2})\, \Delta\eta\, [p_y(\eta_{n+1/2}) + H(p_x(\eta_{n+1/2}), x(\eta_n))]
\end{aligned} \tag{4.5}$$

Then taking the canonical transformation (2.28), $(x, p_x; y, p_y) \mapsto (a, p_a; \sigma, p_\sigma)$, of the variables in the Feynman integral (4.5), we obtain

$$\langle x'', t'' | x', t' \rangle$$

$$= \int \prod_{k=0}^{n-1} \frac{dp_a(\eta_{k+1/2})}{2\pi} \frac{dp_\sigma(\eta_{k+1/2})}{2\pi} da(\eta_k)\, d\sigma(\eta_k)$$

$$\times \prod_{j=1}^{n-1} \delta\left(\frac{\sigma(\eta_j) + p_a(\eta_{j+1/2})}{p_\sigma(\eta_{j+1/2}) - [A^2 p_\sigma(\eta_{j+1/2})]^{1/3}} - [t_0 + \Delta\eta j]\right)$$

$$\times \delta\left(-\frac{1}{2} p_\sigma^2(\eta_{j+1/2}) + \frac{3}{4} [A p_\sigma^2(\eta_{j+1/2})]^{2/3} + H(p_a(\eta_{j+1/2}), a(\eta_j), p_\sigma(\eta_{j+1/2}))\right)$$

$$\times \exp i \sum_{l=0}^{n-1} [p_a(\eta_{l+1/2})(a_{l+1} - a_l) + p_\sigma(\eta_{l+1/2})(\sigma_{l+1} - \sigma_l)]$$

$$\times \delta(a(\eta_0) + p_\sigma(\eta_0) - x_0)\delta\left(\frac{\sigma(\eta_0) + p_a(\eta_{1/2})}{p_\sigma(\eta_{1/2}) - [A^2 p_\sigma(\eta_{1/2})]^{1/3}} - t_0\right)$$

$$\times \exp i[\omega^0(t_n) - \omega^0(t_0)] \tag{4.6}$$

The number of momentum and configuration integrations does not coincide, so we introduced δ-functions at points x_0, y_0 for a matching: $\delta(x(\eta_0) - x_0)\delta(y(\eta_0) - y_0)$. We supposed in the limit $n \to \infty$ that the points of integration over momenta tend to the left to the points of integration over the coordinates. Because the Liouville measure is conserved under a canonical transformation, we used this property. Then, the exponent is an integral of 1-form ω^1. Under canonical mapping an exact form ω^0, (2.31), arises. We integrate over nondiscontinuous trajectories, so the exponential factor defined in the limits of the time interval appears in the last line in (4.5).

Integrating over $a(\eta_0)$ and $\sigma(\eta_0)$, we get

$$\langle x'', t'' | x', t' \rangle$$

$$= \int Da\, Dp_a\, D\sigma\, Dp_\sigma\, \delta(\varphi_0)\delta(\chi) | p_\sigma(\eta_{1/2}) - [A^2 p_\sigma(\eta_{1/2})]^{1/3} |$$

$$\times \exp i[\omega^0(t_n) - \omega^0(t_0)] \tag{4.7}$$

where χ is the gauge. The Faddeev–Popov determinant in our case is unity. The measure is written in a standard form. One should not overlook a mismatch in the number of momentum and configuration integrations (there is one unpaired momentum integration).

Using the Faddeev–Popov method, we are to multiply the measure at the boundary points of the path integral in the factor in (4.7). The factor in (4.7) is defined at the point $\eta_{1/2}$ or $\eta_{n-1/2}$. It depends on the choice in the limit $n \to \infty$ of whether momenta points (intermediate points) tend to the left or to the right of adjacent coordinate points.

An attractive description of systems with constraints is provided by the Becchi–Rouet–Stora–Tyutin (BRST) formalism. Rather than reducing the phase space, it extends it by adding Grassmann dimensions. The corresponding extended phase space (superspace) Batalin–Fradkin–Vilkovisky (BFV) functional integral corresponds to an unconstrained Hamiltonian system. There is a general method for making this transition [see, e.g., Halliwell (1988) for systems with finite degrees of freedom].

The total action is

$$S_T = \int_{t'}^{t''} d\eta \left(p_a \frac{da}{d\eta} + p_\sigma \frac{d\sigma}{d\eta} + \Pi \frac{dN}{d\eta} + \bar{\rho} \frac{dc}{d\eta} + \bar{c} \frac{d\rho}{d\eta} - \{\bar{\rho} N + \bar{c}\chi, \Omega\} \right) \tag{4.8}$$

One adds four anticommuting Grassmann variables c, $\bar{c}$, ρ, $\bar{\rho}$. Here Π is a canonically conjugate variable to N, playing the role of Lagrange multiplier, χ is an arbitrary function from the gauge-fixing condition $\dot{N} = \chi(a, p_a, \sigma, p_\sigma, N)$, $\Omega \equiv cH + \rho\Pi$ is the BRST charge, and $\{\cdot, \cdot\}$ is the super-Poisson bracket. The Lie algebra is supplemented by the following relations between generators in the superspace:

$$\{\bar{c}, \rho\} = 1, \qquad \{\bar{\rho}, c\} = 1, \qquad \{N, \Pi\} = 1 \tag{4.9}$$

The BRST charge generates the BRST flow of some value F:

$$\delta F = \{F, \Lambda\Omega\} \tag{4.10}$$

where Λ is a constant anticommuting parameter. In our case we get the BRST transformation

$$\begin{aligned} &\delta q = \Lambda c, \qquad \delta p = -\Lambda c, \qquad \delta N = \Lambda\rho, \qquad \delta\Pi = 0 \\ &\delta c = 0, \qquad \delta\rho = 0, \qquad \delta\bar{c} = -\Lambda c, \qquad \delta\bar{\rho} = -\Lambda H \end{aligned} \tag{4.11}$$

The classical trajectory is given by the condition of the extremum of S_T, (4.9):

$$\begin{aligned} \dot{q} &= N\frac{\partial H}{\partial p} + \bar{c}c\left\{\frac{\partial\chi}{\partial p}, H\right\} + \frac{\partial\chi}{\partial p}\Pi + \bar{c}\rho\frac{\partial^2\chi}{\partial p\partial N} \\ \dot{p} &= -N\frac{\partial H}{\partial q} - \bar{c}c\left\{\frac{\partial\chi}{\partial q}, H\right\} - \frac{\partial\chi}{\partial q}\Pi - \bar{c}\rho\frac{\partial^2\chi}{\partial q\partial N} \\ \dot{\Pi} &= -H - \bar{c}c\left\{\frac{\partial\chi}{\partial N}, H\right\} - \frac{\partial\chi}{\partial N}\Pi - \bar{c}\rho\frac{\partial^2\chi}{\partial N^2} \end{aligned} \tag{4.12}$$

$$\dot{N} = \chi, \qquad \dot{\bar{\rho}} = -c\{\chi, H\}, \qquad \dot{\rho} = c\{\chi, H\} + \rho\frac{\partial\chi}{\partial N}$$

$$\dot{\bar{c}} = -\bar{\rho} - \bar{c}\frac{\partial \chi}{\partial N}, \qquad \dot{c} = \rho$$

with boundary conditions

$$p\delta q|_{t0}^{tn} = \Pi\delta N|_{t0}^{tn} = \bar{\rho}\delta c|_{t0}^{tn} = \bar{c}\delta\rho|_{t0}^{tn} = 0 \tag{4.13}$$

We obtain solutions of the system (4.13) if we choose a gauge $\chi = 0$:

$$\dot{q} = N\frac{\partial H}{\partial p}, \qquad \dot{p} = -N\frac{\partial H}{\partial q}, \qquad H = 0, \qquad N = N_0$$

$$\bar{\rho} = 0 \qquad \rho = \rho_0, \qquad \bar{c} = 0, \qquad c = \rho_0 t + c_0 \tag{4.14}$$

The Fermi variables are separated from the Bose ones. For the latter variables we have found the Hamiltonian equations and the constraint.

Now the path integral is defined in superspace:

$$\langle x(t'')\,|\,x(t')\rangle = \int D\mu \exp iS_T \tag{4.15}$$

where the measure is

$$D\mu = Dp_a\, Da\, Dp_\sigma\, D\sigma\, D\Pi\, DN\, D\rho\, D\bar{c}\, D\bar{\rho}\, Dc \tag{4.16}$$

The BFV approach is independent of the choice of gauge-fixing function χ by the Batalin–Vilkovisky theorem. So we may put $\chi = 0$, and the ghosts decouple from the other variables. So, after integration by Grassmann variables (Halliwell, 1988), we obtain for the Green function

$$\langle x(t'')\,|\,x(t')\rangle = \int DN(\eta_n - \eta_0) \int Dp_a\, Da\, Dp_\sigma\, D\sigma \times \exp i\int_{t'}^{t''} d\eta \left[p_a \frac{da}{d\eta} + p_\sigma \frac{d\sigma}{d\eta} - N\varphi_0(p_\sigma, a, p_a)\right] \tag{4.17}$$

If we had started from the canonical transformed system of reference $(x, p_x, \sigma, p_\sigma)$ we would have obtained the expression

$$\langle x(t'')\,|\,x(t')\rangle = \int DN(\eta_n - \eta_0) \int Dp_x\, Dx\, Dp_t\, Dt \times \exp i\int_{t'}^{t''} d\eta \left[p_t \frac{dt}{d\eta} + p_\sigma \frac{dx}{d\eta} - N(p_t + H(p_x, x))\right] \tag{4.18}$$

To compare the obtained expressions for the Green functions, let us carry out the canonical transformation (2.28) of the variables in (4.18). We find

$$\langle x'', t'' | x', t' \rangle = \int Da\, Dp_a\, D\sigma\, Dp_\sigma\, \delta(\varphi_0) \exp i \int_{t'}^{t''} d\eta \left(p_a \frac{da}{d\eta} + p_\sigma \frac{d\sigma}{d\eta} - N\varphi_0 \right) \times \exp i[\omega^0(t_n) - \omega^0(t_0)] \tag{4.19}$$

This differs from the BFV expression for the Green function by the exponential multiplier in (4.19).

Thus we are to redefine the measure of integration in both the Faddeev–Popov and Batalin–Fradkin–Vilkovisky approaches according to the above formulas.

REFERENCES

Burlankov, D. E., Dutyshev, V. N., and Kochnev, A. A. (1984). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **87**, 705.

De Witt, B. S. (1967). *Physical Review*, **160**, 1113.

Efimov, N. V. (1978). *Higher Geometry*, Nauka, Moscow.

Fagundes, H. V. (1982). *Letters in Mathematical Physics*, **6**, 417.

Fagundes, H. V. (1983). *Physical Review Letters*, **51**, 517.

Fagundes, H. V. (1985). *Astrophysical Journal*, **291**, 450.

Fagundes, H. V. (1992a). The smallest universe of negative curvature, preprint IFT-P.051/92.

Fagundes, H. V. (1992b). *General Relativity and Gravitation*, **24**, 199.

Guven, J., and Ryan, M. P., Jr. (1992). *Physical Review D*, **45**, 3559.

Halliwell, J. J. (1988). *Physical Review D*, **38**, 2468.

Landau, L. D., and Lifshitz, E. M. (1974). *Quantum Mechanics. Nonrelativistic Theory*, Nauka, Moscow.

Pavlov, A. (1992). *Physics Letters A*, **165**, 211.

Schmutzer, E., ed. (1980). *Exact Solutions of the Einstein Field Equations*, Deutscher Verlag der Wissenschaften, Berlin.

Slavnov, A., and Faddeev, L. (1988). *Introduction to Quantum Theory of Gauge Fields*, Nauka, Moscow.

Smirnov, V. I. (1974). *Course of Higher Mathematics*, Nauka, Moscow.